Engineering Mathematics

(Th- 03)

(As per the 2020-21 syllabus of the SCTE&VT, Bhubaneswar, Odisha)

First & Second

Semester

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Chapter-1   Determinants and Matrices

Content :

- ➢ **Definition of determinant**
- ➢ **Order of determinant**
- ➢ **Minor & cofactor**
- ➢ **Properties of determinant**
- ➢ **Cramer's rule (two variable)**

Introduction:

System of algebraic equations can be expressed in the form of matrices .

. The values of variables satisfying all the linear equations in the system is called solution of the system of linear equations.

. If the system of linear equations has a unique solution , this unique solution is called **determinant** of solution .

C. Definition :

. A determinant is defined as a function from set of square matrices to the set of real numbers .

. Every square matrix A is associated with a number , called its determinant ,

 denoted by det (A) or |A| or Δ .

. Only square matrices have determinants .

If the linear equations

ax + b = 0

cx + d =0

have the same solution then $\frac{b}{a} = \frac{d}{c}$

Or ad - bc = 0

The expression (ad – bc) is called a determinant and is denoted by the symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ where a, b, c , d are called elements of the determinant . The elements in the horizontal direction form rows and elements in the vertical direction form columns.

The det. Of A is written as |A| and is read as det. of A not modulus of A .

The above determinant has two rows and two columns , so it is called a determinant of 2nd order.

Similarly, $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ Is a 3rd order determinant.

Minors :-

The minor of an element $a_{ij} = M_{ij}$ = the det. Obtained by omitting the ith row and j^{th} column of a det. in which a particular element occurs is called the minor of that element.

Minor of an element in a 3rd Order determinant is a 2nd order determinant. Therefore in the above 3rd determinant Minor of a is $\begin{vmatrix} q & r \\ y & z \end{vmatrix}$.

The Minor of b is $\begin{vmatrix} p & r \\ x & z \end{vmatrix}$ and c is $\begin{vmatrix} p & q \\ x & y \end{vmatrix}$. similarly we can find out the Minors for p, q , r and x , y ,z .

If A = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

And Δ stands for the value of the determinant then

Det (A) = |A| = Δ = ad -bc

And for 3rd order det. ,

If Δ = $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ then expanding along 1st row , we get ,

= a $\begin{vmatrix} q & r \\ y & z \end{vmatrix}$ - b $\begin{vmatrix} p & r \\ x & z \end{vmatrix}$ + c $\begin{vmatrix} p & q \\ x & y \end{vmatrix}$

= a M_{11} − b M_{12} + c M_{13} ,where M_{11}, M_{12} , M_{13} are minors of a, b , c respectively .

In the expansion of the determinant of 3rd order the signs with which the elements are multiplied may be remembered by the following formula.

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Cofactors :-

The cofactor of an element in a determinant is its coefficient in the expansion of the determinant .

It is therefore equal to the corresponding Minor with proper sign. Cofactors are generally denoted by C_{ij} .

C_{ij}= $(-1)^{i+j} M_{ij}$

Wher$e$ C_{ij} and M_{ij} are respectively co-factor and minor of element a_{ij} .

Thus in the determinant, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Cofactor of $a_{11}=(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix}$, Cofactor of $a_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix}$, Cofactor of a_{13} = $(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}$

Similarly we can find out the cofactors of other elements .

d. Properties of Determinant :-

1. The Value of a det. does not change if the rows and columns of a determinant are interchanged

2. If two adjacent rows or columns of a determinant are interchanged then the value of the determinant is changed by sign but the absolute value remains same .

3. If two rows or columns of a determinant are identical then the value of the determinant is zero .

4. If each element of any row or any column of a determinant is multiplied by same factor then the determinant is multiplied by that factor .

5. If every element of any row or column of a determinant can be expressed as sum of two number then the determinant can be expressed as the sum of two determinants.

6. A determinant remains unchanged by adding 'K' times the element of any row or column to corresponding element of any other row or column where 'K' is any number.

e. EXAMPLES :

1. Evaluate $\begin{vmatrix} sin\alpha & -cos\alpha \\ cos\alpha & sin\alpha \end{vmatrix}$

   Solution : $sin^2\alpha + cos^2\alpha = 1$

2. Find the value of $\begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix}$.
   Solution : (4) (2) - (3) (-1) = 8+3 = 11

3. Without expanding prove that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ = (a – b) (b- c) (c- a)

   Solution : Applying $R_2 \to R_2 - R_1 , R_3 \to R_3 - R_1$, we get

   $$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

   Taking factors (b -a) & (c- a) common from R_2 & R_3 , respectively , we get

   $$\Delta = (b\text{ -a })(c\text{- a}) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$$= (b - a)(c - a)[(-b + c)] \text{ (expanding along 1}^{st}\text{ column)}$$
$$= (a - b) (b - c) (c - a) \quad \text{(proved)}$$

4. Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution ; L. .S $= \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$ Applying $c_1 \rightarrow c_1 + c_2 + c_3$

$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0 \text{ (proved)}$

f. Solution of simultaneous linear equations by Cramer's rule (two variables) :

The solution of two equations

$a_1 X + b_1 y = d_1$

$a_2 x + b_2 y = d_2$

are given by X = $\frac{\Delta_x}{\Delta}$, y = $\frac{\Delta_y}{\Delta}$ where $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\Delta_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$ and $\Delta_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

Ēxample : Solve the following by Cramer's Rule

Here $\Delta = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 + 3 = 5$, $\Delta_x = \begin{vmatrix} 2 & -1 \\ 13 & 1 \end{vmatrix} = 2+13 = 15$ and $\Delta_y = \begin{vmatrix} 2 & 2 \\ 3 & 13 \end{vmatrix} = 26 - 6 = 20$

$\therefore$ X = $\frac{\Delta_x}{\Delta} = \frac{15}{5} = 3$, y = $\frac{\Delta_y}{\Delta} = \frac{20}{5} = 4$

Short Questions :

1. Find the cofactors of each element in = $\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 2 & 3 & 5 \\ 3 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$.

3. Find the maximum value of $\begin{vmatrix} sin^2x & sinx.cosx \\ -cosx & sinx \end{vmatrix}$ (W-20)

4. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-b-a \end{vmatrix} = (a+b+c)^3$ (w − 20)

Long questions :

5. Without expanding prove that $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc \ (a-b)(b-c)(c-a)$ .s

6. Without expanding prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

7. Solve the following by Cramer's Rule   (w- 20)

$$2x - 3y = 7$$

$$3x - 2y = 3$$

$\approx 0 \approx$

Chapter-1 MATRICES

Contents :

MATRIX :-   A Matrix is a rectangular array of numbers arranged in rows and columns. If there are 'm' rows and 'n' columns in a matrix then it is called an 'm' by 'n' Matrix or a matrix of **order m× n**. The first letter in m×n denotes the number of rows and the second letter n denotes the number of columns. Generally, the capital letters of English alphabet are used to denote matrices and the actual matrix is enclosed in parentheses.

Hence $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & & \ddots & \vdots\ a_{2n} \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$

is a matrix of order m×n and a_{ij} denotes the element in the ith row and j^{th} column . For example a_{23} is the element in the 2ⁿᵈ row and 3ʳᵈ column .Thus the matrix A may be written as (a_{ij}) where i takes values from 1 to m to represent row and j takes values from 1 to n to represent column.

if m=n then the matrix A is called a square matrix of order n by n .Thus

$A = \begin{bmatrix} a_{11} & a_{12}\ a_{13} \cdots & a_{1n} \\ a_{21}\ \vdots & \ddots & a_{2n}\ \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ is a square matrix of order n.

a . Types of Matrices:

1. **Row matrix**: A matrix with a single row is called a row matrix.

2. **Column Matrix** : A matrix with a single column is called a column matrix.

3**. Square Matrix**: A matrix in which number of rows is equal to number of columns is called a square matrix.

4. **Diagonal matrix**: A square Matrix in which the non-diagonal elements are all zero is called a diagonal matrix

5**. Scalar Matrix**: A diagonal Matrix in which the diagonal elements are all equal is called a scalar matrix.

6. **Unit Matrix** : The square Matrix whose elements on its main diagonal (left top to right bottom) are all unity is called a unit matrix. It is denoted by I and it may be of any order.

7. **Zero matrix**: A matrix in which all the elements are all zero is called a zero matrix.

8. **Singular matrix**: A square matrix whose determinant value is zero is called a singular matrix

9. **Non-singular matrix**: A square matrix whose determinant value is not zero is called a non- singular matrix.

10**. Transpose of matrix :** The transpose of a matrix A is the matrix obtained from A by changing its rows into columns and columns into rows. It is denoted by A das or AT

b. Algebra of Matrices:

Equality of two matrices:

 Two matrices A and B are said to be equal if and only if

1. Order of A and B are same .

2. Each element of A is equal to the corresponding element of B .

Additon of matrices :

The sum of two matrices A &B is the matrix such that each of its elements is equal to the sum of the corresponding elements of A and B . The sum is denoted by A + B. Thus the addition of matrices is defined if they are of same order and is not defined when they are of different orders. A, B & A+B are of same order.

Subtraction of Matrices:— The subtraction Of two matrices A & B of the same order is defined as A- B = A+(-B)

Product of a matrix and a scalar:-

The product of a scalar m and a matrix A is denoted by a mA, is the matrix each of whose elements is m times the corresponding element of A.

Ex. If $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \\ 3 & -2 & 1 \end{pmatrix}$ then $3A = \begin{pmatrix} 3 \times 2 & 3 \times 1 & 3 \times 3 \\ 3 \times -1 & 3 \times 0 & 3 \times 4 \\ 3 \times 3 & 3 \times -2 & 3 \times 1 \end{pmatrix} = \begin{pmatrix} 6 & 3 & 9 \\ -3 & 0 & 12 \\ 9 & -6 & 3 \end{pmatrix}$

Product of two matrices:- The product of two matrices A & B **(where the number of columns in A is equal to the number of rows in B)** is the matrix AB whose element in the ith row and j^{th} column is the sum of the products formed by multiplying each element in the ith row of A and the corresponding element in the jth column of B.

Let A be an (m× k) matrix and B be a (k ×n) matrix. The product of A and B, denoted by AB is the (m×n) matrix with $(i, j)^{th} entry$ equal to the sum of the products of a corresponding elements from ith row of A and Jth column of B.

For Example : 1. if A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, B = $\begin{pmatrix} x & y \\ u & v \end{pmatrix}$ then AB = $\begin{pmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{pmatrix}$

Ex.- 2.   if A = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, B = $(x \ y \ z)$ then AB = $\begin{pmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{pmatrix}$

Ex-3.   if A = $(a \ b \ c)$, B = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then AB = $(ax + by + cz)$

Properties :

   (I)   The multiplication of matrices is not necessarily commutative i.e, if A and B are two matrices then AB is not equal to BA .

(II)   The multiplication of matrices is associative i.e, if A,B,C are three matrices then (AB)C = A(BC), Provided the products are defined.

(III)   The identity matrix for multiplication for the set of all square matrices of a given order is the unit matrix of the same order

(IV)   Let A and B be 2 matrices such that the product AB is defined. Then A = 0 or B=0 or A=0= B always implies that AB equal to 0. conversly AB=0 does not always imply that A =0 or B= 0 or A=0= B.

(V)   The cancellation law does not hold for matrix multiplication, i.e CA=CB does not necessarily imply A equal to B.

(VI)   The distributive laws hold good for matrices. If A, B & C are three matrices then A (B+C)= AB + AC , (A+B)C= AC + BC provided the addition and multiplication in above equations are defined .

TRANSPOSE OF A MATRIX :

Transpose of a m×n matrix A is the matrix of order n×m obtained by interchanging the rows and columns 0f A. The transpose of a matrix A is written as A' or A^T.

For example : if A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then A^T = $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Adjoint of a matrix : The adjoint of a matrix A is the transpose of the matrix obtained by replacing each element a_{ij} in A by its cofactor c_{ij} . The adjoint of A is written as adj A . Thus if

A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then adj A = $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

e . INVERSE OF MATRIX :   If A is a non-singular matrix, then it's inverse matrix A^{-1} is defined as

   $A^{-1} = \frac{adjA}{|A|}$ Where |A| is the determinant of matrix A .

Ex : 4 Find inverse of the matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.

Solution : Expanding along R_1, $|A| = 0 - 1 \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = -(1-9) + 2(1-6) = 8-10 = -2 \neq 0$

So, the inverse of A exists. Let C_{ij} and M_{ij} are cofactors and Minors of A.

Hence $C_{ij} = (-1)^{i+j} M_{ij}$

Cofactor of 0, $C_{11} = (-1)^{(1+1)} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = +(2-3) = -1$

Cofactor of 1, $C_{12} = (-1)^{(1+2)} \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -(1-9) = 8$

Cofactor of 2, $C_{13} = (-1)^{(1+3)} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = +(1-6) = -5$

Similarly cofactor of 1, $C_{21} = (-1)^{(2+1)} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -(1-2) = 1$

Cofactor of 2, $C_{22} = (-1)^{(2+2)} \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = +(0-6) = -6$

cofactor of 3, $C_{23} = (-1)^{(2+3)} \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -(0-3) = 3$

cofactor of 3, $C_{31} = (-1)^{(3+1)} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = +(3-4) = -1$

cofactor of 1, $C_{32} = (-1)^{(3+2)} \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -(0-2) = 2$

cofactor of 1, $C_{33} = (-1)^{(3+3)} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = +(0-1) = -1$

we have

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adjA}{|A|} = \frac{\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}}{-2} = \begin{bmatrix} -1/2 & 1/2 & -1/2 \\ 4 & -3 & 1 \\ -\frac{5}{2} & 3/2 & -1/2 \end{bmatrix} \quad (\text{ANS})$$

Solution of system of Linear equations by Matrix method : -

Suppose we have the following system of equations

$a_1 x + b_1 y + c_1 z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_2y + c_3z = d_3$

Where A = $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, x = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and B = $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

The equation in the matrix form becomes AX = B $\Rightarrow$ X = $A^{-1}B$, if $A \neq 0$

EX. -5 Solve the following system of equations by Matrix method

$$x - y + z = 4 \; , \; 2x + y - 3z = 0 \; , \; x + y + z = 2$$

Solution : Here A = $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, B = $\begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$

$|A|$ = 1(1 + 3) − (-1) (2 + 3) + 1 (2- 1) = 4 + 5 + 1 = 10 ≠ 0

$$c_{11} = +(1+3) = 4, \quad c_{12} = -(2+3) = -5 \quad c_{13} = = (2-1) = 1$$

c_{21} = - (-1 -1) = +2 c_{22} = + (1 -1) = 0 c_{23} = - (1 +1) = -2 ,

$c_{31} = +(3-1) = 2$, $c_{32} = -(-3-2) = = 5$, $c_{33} = +(1+2) = 3$

Cofactor Matrix of A = $\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$

So, AdjA = $\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

X = $A^{-1}B \Rightarrow X = \frac{AdjA}{|A|} \cdot B = \frac{\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}}{10} \cdot \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{bmatrix} \frac{2}{5}.4 + \frac{1}{5}.0 + \frac{1}{5}.2 \\ \frac{-1}{2}.4 + 0.0 + \frac{1}{2}.2 \\ 1\frac{11}{10}.4 - \frac{1}{5}.0 + \frac{3}{10}.2 \end{bmatrix} = \begin{bmatrix} \frac{8}{5} + \frac{2}{5} \\ -2 + 1 \\ \frac{2}{5} + \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow$ X = 2 , y = -1 , z = 1   (ans)

SHORT QUESTIONS :-

1. Find x & y if $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2. If $\begin{bmatrix} 3 & 4 & 2 \end{bmatrix}$ B = $\begin{bmatrix} 2 & 1 & 0 & 3 & 6 \end{bmatrix}$ then find the order of B .   (2010-w , 2014 w)

3. Write down the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ if a_{ij} = 2i +3j   (2008w)

4. Find the adjoint of the matrix $\begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$.

5. If A = $\begin{bmatrix} 2 & 4 \\ 3 & 13 \end{bmatrix}$ and I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then find A- αI .   (2012-W)

   LONG QUESTIONS :

   1. Find the adjoint of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ (2012 W)

   2. Find the inverse of the followings

      (i) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ (2008 W) (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ (2014 W) (iii) $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$ (2010w)

   3. Solve the following equations by Matrix method   (2010 W)
      X – y + z = 4 , 2x + y -3z = 0 , x + y + z = 2

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Chapter- 2 Trigonometry

Objectives :

- **Preliminary ideas , Trigonometry functions and identities**
- **Trigonometry ratios**
- **Compound angles, multiple angles, submultiple angles (0nly formulae)**
- **Problems on above**
- **Definition of inverse circular function**
- **Study its properties**
- **Problems**
  - **Short questions and Long questions**

  Introduction : The word trigonometry derived from greek words trigono and metron , means the measurement of angles in a triangle. Trigonometry is that branch of mathematics which deals with the relationship between the sides and angles of a triangle .

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A. :

Measurement of an angle : -

There are several units for measuring angles .Two most commonly used units are degree measure and radian measure.

1. **Degree measure :**

If a rotation from the initial side to terminal side is (1 /3 60) th of a revolution, the angle is said to have a measure of 1 degree written as 1° . A degree is divided into 60 minutes and a minute is divided into 60 seconds .

2. Radian measure :-

There is another unit for measurement of an angle called the radian measure. Angle subtended at the center by an arc of length one unit in a unit circle (i.e circle of radius 1) is said to have a measure of 1 radian .

If in a circle of radius r an Arc of length l subtends an angle of **P** radians then l= r× **P**

Radian measure = π/180× degree measure

Degree measure = 180/π × radian measure

Trigonometric ratios of angles of any magnitude and sign convention :-

In Cartesian coordinate system X axis and y axis divide the plane into four quadrants .

Case 1 :- When **P** lies in the 1st quadrant , i.e   0° < **P** < 90°

Then sin**P** = p/h= +ve

cos**P**= b/h= x/r is +ve

tan**P**= p/b = x/y is +ve

cot**P**= b/p = y/x is +ve

sec**P**= h/b = r/x is +ve

Cosec**P** = h/p = r/y is +ve
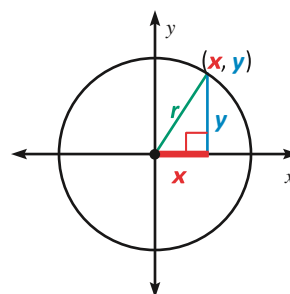
tan**P**= P/b = X / y is +ve

Since in the 1ˢᵗ quadrant x and y both are +ve so all the six ratios are +ve .

CASE-2   when P lies in the 2ⁿᵈ quadrant , i.e 90°< P <180°

Radius vector r Remains positive, y is +ve, x is -ve

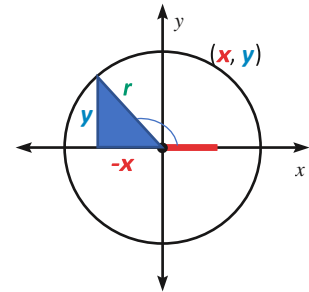 Then sin**P** = p/h= y/r is +ve

Cos**P**= b/h = -x/r is - ve

tan**P**= p/b = y/-x is - ve

Cosec**P**= h/p = r/y is + ve

Sec**P** = h/b= r/-x is -ve

Cot**P**= b/p= -x/y is -ve

Thus in the 2ⁿᵈ quadrant only sin**P** and cosec**P** are positive and all other ratios are negative.

Case-3 When P lies in the 3ʳᵈ quadrant x & y both are -ve .
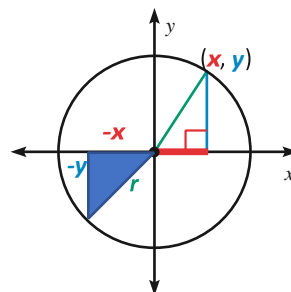
Sin**P**= p/h= -y/r is -ve

Cos**P** = b/h= -x/r is -ve

Tan**P** = p/b=- y/-x is +ve

Cot**P**= b/p= -x/-y is +ve

Sec**P**= h/b = r/-x is -ve

cosec**P**= h/p= r/-y is -ve

Hence in the 3ʳᵈ quadrant only tan**P** and cot**P** are positive and other ratios are negative.

Case-4 when P lies in the 4ᵗʰ quadrant, x is +ve and y is – ve

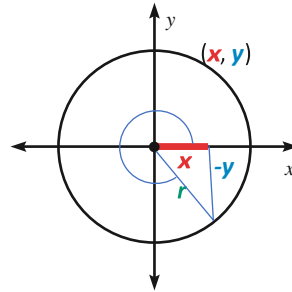**SinP = p/h =** -y/r is -ve

CosP = b/h = x/r is ve**P**

Tan**P** = p/b = -y/x is – ve

CotP= b/p= x/-y is -ve

SecP = h/b = r/x is +ve.

Thus in the 4th quadrant cos**P** and sec**P** are +ve and

Other t-ratios are -ve.

The above result may be briefly written as follows :

ASTC Rule :-

A—> All +ve

S—> Sin +ve

C—> cos +ve

T—> tan +ve

ASTC —> Add Sugar To Coffee

Sign convention for any angle:-

When the revolution of radius vector r is Counter-clockwise then the angle is considered to be positive. when the revolution is clockwise then the angle is considered to be negative .

Note: The sign of radius vector is always positive .

Relation between the trigonometric :-

Reciprocal relation between t – ratios :

Sin**P** = p/h   cosec**P** = h/p

Cos**P** = b/h   sec**P** = h/b

Tan**P** = p/b   cot**P** = b/p

It can be easily seen that sin**P** and Cos**P** are reciprocal of each other . similarly cos**P** and sec**P** are reciprocal of each other and tan**P** and cot**P** are reciprocal of each other .

tan**P** = sin**P**/cos**P**   cot**P** = cos**P**/sin **P**

Relation between trigonometric Ratios : -

Identities:

1. **$Sin^2P + cos^2P = 1$, $sin^2P = 1 - cos^2P$, $cos^2P = 1 - sin^2P$**
   Each of sin^2P and cos^2P < 1.
   So, $-1 \leq sinP \leq 1$ and $-1 \leq cosP \leq 1$
   Thus , domain of sin**P** and cos**P** is the set of all real numbers and range is the interval [-1, 1]
2. **$Sec^2P - tan^2P = 1$, $sec^2P = 1 + tan^2P$ and** $tan^2P = sec^2P - 1$
    (sec**P** + tan**P**)(sec**P** – tan**P**) = 1
    => sec**P** + tan **P** = 1/(sec **P** - tan **P**)
3. **$Cosec^2 P - cot^2 P = 1$,** $cosec^2P = 1 + cot^2 P$ and
    $cot^2P = cosec^2P - 1$
    (Cosec **P** + cot **P**)(cosec**P** - cot**P**) = 1 => cosec **P** + cot**P** = 1/ (cosec **P** - Cot**P**)

Note : $-\infty < tanP < \infty$

❖   **Trigonometric Ratios of Allied Angles**

We might calculate the trigonometric ratios of angles of any value using the trigonometric ratio of allied angles.

$\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ and $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\tan(-\theta) = -\tan\theta$ and $\cot(-\theta) = -\cot\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ and $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

$\cosec(-\theta) = -\cosec\theta$ and $\sec(-\theta) = \sec\theta$

$\sec\left(\frac{\pi}{2} - \theta\right) = \cosec\theta$ and $\cosec\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$ and $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$ and $\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta$

$\sec\left(\frac{\pi}{2} + \theta\right) = -\cosec\theta$ and $\cosec\left(\frac{\pi}{2} + \theta\right) = -\sec\theta$

$\sin(\pi - \theta) = \sin\theta$ and $\cos(\pi - \theta) = -\cos\theta$

$\tan\pi(-\theta) = -\cot\theta$ $\cot(\pi - \theta) = -\cot\theta$

$\cosec(\pi - \theta) = \cosec\theta$ $\sec(\pi - \theta) = -\sec\theta$

$\sin(\pi + \theta) = -\sin\theta$ and $\cos(\pi + \theta) = -\cos\theta$

$\tan(\pi + \Theta) = \cot\theta$ $\cot(\pi + \theta) = \cot\theta$

$\sec(\pi + \theta) = -\sec\theta$ and $\cosec(\pi + \theta) = -\cosec\theta$

$\sin(2\pi - \theta) = -\sin\theta$ and $\cos(2\pi - \theta) = \cos\theta$

$$\tan(2\pi - \theta) = -\tan\theta \text{ and } \cot(2\pi - \theta) = -\cot\theta$$

$$\sec(2\pi - \theta) = \sec\theta \text{ and } \operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec}\theta$$

$$\sin(2\pi + \theta) = \sin\theta \text{ and } \cos(2\pi + \theta) = \cos\theta$$

$$\tan(2\pi + \theta) = \tan\theta \text{ and } \cot(2\pi + \theta) = \cot\theta$$

$$\sec(2\pi + \theta) = \sec\theta \text{ and } \operatorname{cosec}(2\pi + \theta) = \operatorname{cosec}\theta$$

b.Compound Angles :-- When an angle formed as the algebraic sum of two or more angles then the angle is called a compound angle .

Thus (A+B) and (A+B+C) are compound angles ..

Trigonometric Functions of Sum or Difference of Two Angles

A. Addition Formulae: B. Difference Formulae :

(a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$ (b) $\sin(A - B) = \sin A \cos B - \cos A \sin$

c) $\cos(A + B) = \cos A \cos B - \sin A \sin B$ (d) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(e) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ (f) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(g) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$ (h) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(i) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A + B) \cdot \sin(A - B)$

(j) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A + B) \cdot \cos(A - B)$

(k) $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

The products of sin-cos ratios as their sum and difference :

      1. 2 sin A. cos B = sin (A+B) + sin (A-B)

2. $2 \cos A . \sin B = \sin (A+B) - \cos (A-B)$
3. $2 \cos A . \cos B = \cos (A+B) + \cos (A-B)$
4. $2 \sin A . \sin B = \cos (A-B) - \cos (A+B)$

Factorisation of the sum or difference of two sines or cosines (Product Formulae) : -

(a) $\sin C + \sin D = 2\sin \frac{C+D}{2} \cos \frac{C-D}{2}$

(b) $\sin C - \sin D = 2\cos \frac{C+D}{2} \sin \frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos \frac{C+D}{2} \cos \frac{C-D}{2}$

(d) $\cos C - \cos D = -2\sin \frac{C+D}{2} \sin \frac{C-D}{2}$

Trigonometric ratios of multiple angles :-

1. $\sin 2A = 2\sin A . \cos A = 2\tan A / (1+ \tan^2 A)$
2. $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A -1 = 1 - 2\sin^2 A = (1- \tan^2 A)/ (1+ \tan^2 A)$;
3. $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$;
4. $\sin 3A = 3\sin A - 4 \sin^3 A$
5. $\cos 3A = 4\cos^3 A - 3 \cos A$
6. $\tan 3 A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Trigonometric ratios of sub-multiple angles :-

1. $\sin A = 2\sin\frac{A}{2} . \cos \frac{A}{2}$
2. $2 \cos^2 A/2 = 1 + \cos A ,\quad 2\sin^2 A/2 = 1 - \cos A$
3. $\tan A = (2\tan\frac{A}{2})/ (1-\tan^2 \frac{A}{2})$

❖ . Important Trigonometric Ratios

(a) $\sin n\pi = 0; \cos n\pi = (-1)^n; \tan n\pi = 0$ where $n \in Z$

(b) $\sin 15° $ or $ \sin\dfrac{\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° $ or $ \cos\dfrac{5\pi}{12};$

$\cos 15° $ or $ \cos\dfrac{\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° $ or $ \sin\dfrac{5\pi}{12}$

$$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3} = \cot 75°$$

$$\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} = \cot 15°$$

(c) $\sin\dfrac{\pi}{10}$ or $\sin 18° = \dfrac{\sqrt{5}-1}{4}$ & $\cos 36°$ or $\cos\dfrac{\pi}{5} = \dfrac{\sqrt{5}+1}{4}$

PROBLEMS :

1. If (A + B) = 45⁰, find the value of (1 + tan A) (1 + tan B).

Solution: Given that A+ B = $45°$

     Taking tan in both sides , tan (A +B) = $\tan 45°$

$\Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A.\tan B}$ = 1 $\Rightarrow$ tanA + tan B = 1- tanA. tanB

$\Rightarrow$ tanA + tan B + tanA .tan B = 1

Adding 1 on both sides , $\Rightarrow$ 1+ tanA + tan B + tanA .tan B = 1 +1 =2

$\Rightarrow$ (1+ tanA) + tan B (1 + tanA)= 1 +1 = 2 ,

$\Rightarrow (1 + \tan A)(1 + \tan B) = 2$ (ans)

2. If tan α = $\dfrac{1}{2}$, tan β = $\dfrac{1}{3}$ then find the value of (α+β) .

Solution : $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = = \dfrac{5/6}{5/6} = 1 = \tan 45^\circ$

$\Rightarrow (\alpha + \beta) = 45^\circ$ (ans)

3. Find the value of $\sin 105^\circ . \cos 5^\circ$.

Solution : $\sin 105^\circ . \cos 5^\circ = \dfrac{1}{2}$ (2 $\sin 105^\circ . \cos 5^\circ$) = $\dfrac{1}{2}$ (2 $\sin$ (90 +5°) . $\cos 5^\circ$)

4. If $\dfrac{1+\sin A}{\cos A} = \sqrt{2} + 1$, then find the value of $\dfrac{1-\sin A}{\cos A}$.

5.

INVERSE TRIGONOMETRIC FUNCTIONS

BASIC CONCEPTS

INVERSE CIRCULAR FUNCTIONS

1. y = sin-1x   iff   x=siny

2. y = cos-1x   iff   x=cosy

3. y = tan-1x   iff   x= tany

4. y = cot-1 x   iff   x=coty

5. y = cosec-1x iff   x=cosecy

6. y = sec-1x   iff   x=secy

.

PROPERTY – I

(i) $\sin^{-1}x + \cos^{-1}x = \pi/2$, for all $x \in [-1,1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \pi/2$, for all $x \in R$

(iii) $\sec^{-1} x + \csc^{-1} x = \pi/2$ for all $x \in (-\infty,-1] \cup [1, \infty)$

PROPERTY – II

(i) $\sin^{-1}x = \csc^{-1}\left(\frac{1}{x}\right)$

(ii) $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right)$

(iii) $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$

PROPERTY -III

(i) $\cos^{-1}(-x) = \pi-\cos^{-1}(x)$ for all $x \in [-1,1]$

(ii) $\sec^{-1}(-x) = \pi-\sec^{-1}x$, for all $x \in (-\infty,-1] [1, \infty)$

(iii) $\cot^{-1}(-x) = \pi-\cot^{-1}x$, for all $x \in R$

(iv) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$

(v) $\tan^{-1}(-x) = -\tan^{-1}x$,   for all $x \in R$

(vi) $\csc^{-1}(-x) = -\csc^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

 PROPERTY – IV

(i) $\sin(\sin^{-1}x)=x$,   for all $x \in [-1, 1]$

(ii) $\cos(\cos^{-1}x)=x$, for all $x \in [-1, 1]$

 (iii) $\tan(\tan^{-1}x)=x$, for all $x \in R$

(iv) $\csc(\csc^{-1}x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) sec(sec−1x)=x, for all x ∈ (−∞, −1] ∪ [1, ∞)

FORMULAS :

i) tan−1x + tan−1y= tan−1 (x +y)/1-xy , xy <1

ii) tan−1x - tan−1y = tan−1 (x +y)/1-xy , xy > -1

iii) 2 tan−1x = tan−1 $(\frac{2x}{1-x^2})$, |x | < 1

iv) 2tan−1x = sin−1$(\frac{2x}{1+x^2})$, |x| < 1

v) 2tan−1x = cos−1 $(\frac{1-x^2}{1+x^2})$, $x \geq 0$

vi) sin−1x + sin−1y= sin−1 (x . $\sqrt{1-y^2}$ + y . $\sqrt{1-x^2}$)

vii) sin−1x - sin−1y= sin−1(x . $\sqrt{1-y^2}$ − y . $\sqrt{1-x^2}$)

viii) cos−1x + cos−1 y = cos−1 (xy - $\sqrt{1-x^2}$. $\sqrt{1-y^2}$)

ix) cos−1x - cos−1 y = cos−1 (xy + $\sqrt{1-x^2}$. $\sqrt{1-y^2}$)

x) tan−1 x+ tan−1y+ tan−1z = tan−1[$\frac{x+y+z-xyz}{1-xy-yz-zx}$] if x >0, y >0, z > 0 & xy+yz+zx <1

Note ☺
(i) if tan−1 x + tan−1y+ tan−1z = π then x+y+z = xyz
(ii) If tan−1 x+tan−1y+tan−1z= π/2 then xy+yz+zx=1
REMEMBER THAT :

(i) sin−1 x+sin−1 y+sin−1 z= 3π/2 => x= y= z= 1

(ii) $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$ $x=y=z=-1$

(iii) $\tan^{-1}1 + \tan^{-1}2 + 2\tan^{-1}3 = \tan^{-1}1 + \tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/2$

Short questions:

    (a) Prove that $4\cos105^{\circ}.\cos15^{\circ} + 1 = 0$.   (w-19)

6. If $\cos A = \dfrac{1}{2}$, $\cos B = 1$, then prove that $\tan\dfrac{A+B}{2} . \tan\dfrac{A-B}{2} = \dfrac{1}{3}$.

7. If $\sin\alpha = \dfrac{15}{17}$ and $\cos\beta = \dfrac{12}{13}$ where α, β are acute angles, find the value of $\sin(\alpha+\beta)$.

8. If α and β lie in the first and second quadrants respectively and if $\sin\alpha = \dfrac{1}{2}$, $\sin\beta = \dfrac{1}{3}$ then find the value of $\sin(\alpha + \beta)$.   (w-19)

9. If $\dfrac{1+\sin A}{\cos A} = \sqrt{2} + 1$, then find the value of $\dfrac{1-\sin A}{\cos A}$.

10. Find $\sin105^{\circ}.\cos5^{\circ}$.

11. Find the value of $\dfrac{\tan15^{0}}{1-\tan^{2}15}$.   (w-19)

12. Find the value of $\sin70^{\circ}(4\cos^{2}20^{\circ}-3)$.   (w-20)
13. Find the value of $\sec^{2}(\tan^{-1}2) + \csc^{2}(\cot^{-1}3)$ (w-19)

    LONG QUESTIONS :

1. IF $A+B+C = \pi$, PROVE THAT
    i. $\sin 2A + \sin 2B + \sin 2C = 4\sin A.\sin B.\sin$.   (w-19)

ii. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cdot \cos B \cdot \cos C$

2. prove that $\tan 37\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$ (w-19)

3. Prove that

   a. $4\sin A \cdot \sin(60 - A) \cdot \sin(60 + A) = \sin 3A$

   b. $\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ = \frac{3}{16}$.   (w-20)

4. Find the value of $\sin^{-1}\frac{1}{\sqrt{5}} + \cos^{-1}\frac{3}{\sqrt{10}}$ (w- 20)

5. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then prove that x+y+z = xyz .
(w -20)

INVERSE TRIGONOMETRIC FUNCTIONS

BASIC CONCEPTS
INVERSE CIRCULAR FUNCTIONS

1. $y = \sin^{-1}x$ iff $x = \sin y$
2. $y = \cos^{-1}x$ iff $x = \cos y$
3. $y = \tan^{-1}x$ iff $x = \tan y$
4. $y = \cot^{-1} x$ iff $x = \cot y$
5. $y = \csc^{-1}x$ iff $x = \csc y$
6. $y = \sec^{-1}x$ iff $x = \sec y$

.

PROPERTY – I
(i) $\sin^{-1}x + \cos^{-1}x = \pi/2$,   for all $x \in [-1,1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \pi/2$,   for all $x \in R$

(iii) $\sec^{-1} x + \csc^{-1} x = \pi/2$ for all $x \in (-\infty, -1] \cup [1, \infty)$

PROPERTY – II

    (i) $\sin^{-1}x = \csc^{-1}\left(\frac{1}{x}\right)$

    (ii) $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right)$

    (iii) $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$

PROPERTY -III
(i) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ for all $x \in [-1,1]$
(ii) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, for all $x \in (-\infty, -1] [1, \infty)$
(iii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, for all $x \in R$
(iv) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$
(v) $\tan^{-1}(-x) = -\tan^{-1}x$,   for all $x \in R$
(vi) $\csc^{-1}(-x) = -\csc^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

PROPERTY – IV
(i) $\sin(\sin^{-1}x) = x$,   for all $x \in [-1, 1]$
(ii) $\cos(\cos^{-1}x) = x$,   for all $x \in [-1, 1]$
 (iii) $\tan(\tan^{-1}x) = x$,   for all $x \in R$
(iv) $\csc(\csc^{-1}x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
 (v) $\sec(\sec^{-1}x) = x$,   for all $x \in (-\infty, -1] \cup [1, \infty)$

FORMULAS :

i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}(x+y)/1-xy$, $xy < 1$

ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}(x+y)/1-xy$, $xy > -1$

iii) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $|x| < 1$

iv) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $|x| < 1$

v) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \geq 0$

vi) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x.\sqrt{1-y^2} + y.\sqrt{1-x^2})$

vii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x.\sqrt{1-y^2} - y.\sqrt{1-x^2})$

viii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1-x^2}.\sqrt{1-y^2})$

ix) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}(xy + \sqrt{1-x^2}.\sqrt{1-y^2})$

x) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if x >0, y >0, z > 0 & xy+yz+zx <1

Note ☺
(i)   if $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \prod$ then $x+y+z = xyz$
(ii)   If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$ then $xy+yz+zx = 1$
REMEMBER THAT :

(i) $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = 3\pi/2 \Rightarrow x = y = z = 1$
(ii) $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\prod$ $x=y=z=-1$
(iii) $\tan^{-1}1 + \tan^{-1}2 + 2\tan^{-1}3 = \tan^{-1}1 + \tan^{-1}(1/2) + \tan^{-1}(1/3) = \prod/2$

Chapter -3

CO-ORDINATE GEOMETRY IN TWO DIMENSIONS

OBJECTIVES

- ❖ **Introduction**
- ❖ **Distance formula, division formula (both internal & external)**
- ❖ **Slope of line & angle between two lines , condition of parallelism & perpendicularity**
- ❖ **Different forms of St. line (i). one point form (ii). Two point form (iii). Slope form (iv). Intercept form   (v) perpendicular form**
- ❖ **Equation of a line passing through a point & (i). parallel to a line (ii) perpendicular to a line**
- ❖ **Equation of a line passing through the intersection of two lines**
- ❖ **Distance of a point from a line**
- ❖ **Problems based on above**

A . INTRODUCTION :

The method of finding the , position of a point in a plane very precisely was introduced by the French Mathematician and Philosopher, Rene Descartes (1596-1650).

In this, a point in the plane is represented by an ordered pair of numbers, called the Cartesian co-ordinates of a point .
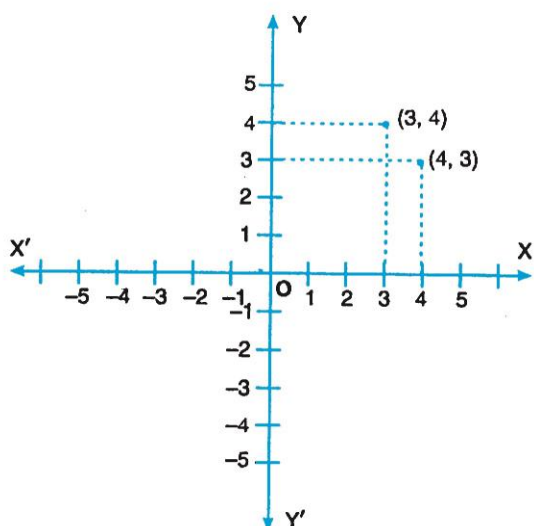
COORDINATE SYSTEM :

The position of a point in a plane is fixed w.r.t. to its distances from two axes of reference, which are usually drawn by the two graduated number lines XOX′ and YOY′, at right angles to each other at O.

The number plane (Cartesian plane) is divided into four quadrants by these two perpendicular axes called the ***x*-axis** (horizontal line) and the ***y*-axis** (vertical line). These axes intersect at a point called the **origin**. The two axes together are calld **rectangular co-ordinate system** .The position of any point in the plane can be represented by an ordered pair of numbers (x, y). These ordered pairs are called the **coordinates** of the point.

It may be noted that, the positive direction of x-axis is taken to the right of the origin O, OX and the negative direction is taken to the left of the origin O, i.e., the side OX'. Similarly, the portion of y-axis above the origin O, i.e., the side OY is taken as positive and the negative direction is taken to the below the origin O, i.e., the side Oy'.

CO-ORDINATES OF A POINT :

The position of a point is given by two numbers, called co-ordinates which refer to the distances of the point from these two axes. By convention the first number , the **x -co-ordinate (or abscissa)**, always indicates the distance from the y-axis and the second number, the **y-coordinate (or ordinate)** indicates the distance from the x-axis .

In general, co-ordinates of a point P(x, y) imply that distance of P from the y-axis is x units and its distance from the x-axis is y units.

You may note that the co-ordinates of the origin O are (0, 0). The y co-ordinate of every point on the x-axis is 0 and the x co-ordinate of every point on the y-axis is 0.

In general, co-ordinates of any point on the x-axis to the right of the origin is (a, 0) and that to left of the origin is (–a, 0), where 'a' is a non-zero positive number.
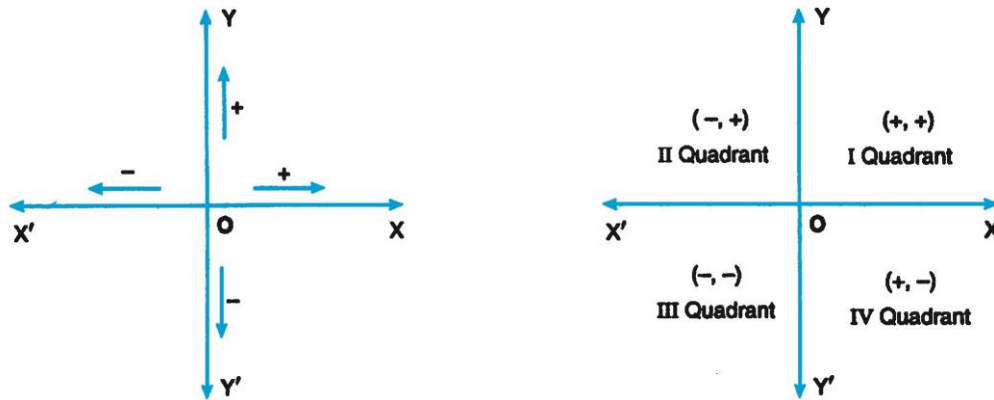
Similarly, y co-ordinates of any point on the y-axis above and below the x-axis would be (0, b) and (0, –b) respectively where 'b' is a non-zero positive number.

You may also note that the position of points (x, y) and (y, x) in the rectangular, coordinate system is not the same.
For example position of points (3, 4) and (4, 3) are shown in above Figure .
It is clear from the point A(3, 4) that its x co-ordinate is 3 and the y co-ordinate is 4. Similarly x co-ordinate and y co-ordinate of the point B(4, 3) are 4 and 3 respectively.

QUADRANTS :

The co-ordinates of all points in the first quadrant are of the type (+, +) (See Fig.)

Any point in the II quadrant has x co-ordinate negative and y co-ordinate positive (–, +), Similarly, in III quadrant, a point has both x and y co-ordinates negative (–,–) and in IV quadrant, a point has x co-ordinate positive and y co-ordinate negative (+,–).

For example :

(a) P (5, 6) lies in the first quadrant as both x and y co-ordinates are positive.

(b) Q (–3, 4) lies in the second quadrant as its x co-ordinate is negative and y co-ordinate is positive.

(c) R (–2, –3) lies in the third quadrant as its both x and y co-ordinates are negative.

(d) S(4, –1) lies in the fourth quadrant as its x co-ordinate is positive and y coordinate is negative

B. DISTANCE BETWEEN TWO POINTS :-

The distance between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane is the length of the line segment PQ.

From P, Q draw PL and QM perpendicular on the x-axis and PR perpendicular on QM.

Then OL= x_1 , OM= x_2, PL= y_1 QM = y_2
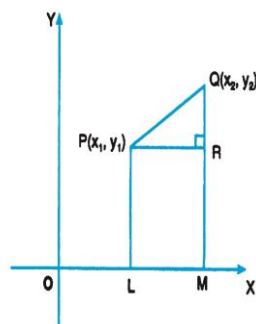
PR = LM = OM – OL = $x_2 - x_1$

QR = QM – RM = QM – PL = y_2 - y_1

Since PQR is a right angled triangle

$$PQ^2 = PR^2 + QR^2$$

$$=(x_2 - x_1)^2 + (Y_2 - Y_1)^2$$

$$PQ = \sqrt{(x_2 - x_1)^2 + (Y_2 - Y_1)}$$

Therefore,

Distance between two points $= \sqrt{(\text{differnce of abcissa})^2 + (\text{difference of ordinate})^2}$

Corollary : The distance of the point (x_1, y_1) from the origin is $\sqrt{x_1^2 + y_1^2}$.

Example -1

Find the distance between two points in each of the following cases

    a) $p(6,8)$ and $Q(-9,-12)$ b) $A(-6,1)$ & $B(-6,-11)$

    Solution : a) By using distance formula, $\mathbf{PQ} = \sqrt{(-9-6)^2 + (-12-8)^2} =$
    $\sqrt{(-15)^2 + (-20)^2} = \sqrt{(225+400)} = \sqrt{625} = 25$ **units**

    b) AB $= \sqrt{\{-6-(-6)\}^2 + (-12-8)^2} = \sqrt{0 + (-20)^2} = \sqrt{400} = 20$ **Units**

SECTION FORMULA :

1.Internal division :

        **Co-ordiantes of a point which divides the line segment joining the points (x_1,
y_1) and (x_2, y_2) in the ratio m : n internally are** $(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n})$**.**

2.External division :

        **Co-ordiantes of a point which divides the line segment joining the points (x_1,
y_1) and (x_2, y_2) in the ratio m : n externally are** $(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n})$**.**

Mid- Point Formula :

  The co-ordinates of the mid-point of the line segment joining two points (x_1, y_1) and (x_2, y_2) can be obtained by taking m = n in the section formula above.

Putting m = n in (1) above, we have

  The co-ordinates of the mid-point joining two points (x_1, y_1) and (x_2, y_2) are:
$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2} \right)$.

Example : Find the co-ordinates of a point which divides the line segment joining each of the following points in the given ratio:

    (a) $(2, 3)$ and $(7, 8)$ in the ratio 2 : 3 internally.

(b) (–1, 4) and (0, –3) in the ratio 5 : 4 externally.

Solution: (a) Let A(2, 3) and B(7, 8) be the given points.

Let P(x, y) divide AB in the ratio 2 : 3 internally.

Using section formula, we have

$$x = \frac{2 \times 7 + 3 \times 2}{2 + 3} = \frac{20}{5} = 4$$

$$\text{and } y = \frac{2 \times 8 + 3 \times 3}{2 + 3} = \frac{25}{5} = 5$$

∴ P(4, 5) divides AB in the ratio 2 : 3 internally.

(b) Let A(–1, 4) and B(0, 3) be the given points.

Let P(x, y) divide AB in the ratio 5: 4 externally.

Using section formula, we have

$$x = \frac{5 \times (0) - 4 \times (-1)}{5 - 4} = \frac{0 + 4}{1} = 4$$

$$\text{and } y = = \frac{5 \times (3) - 4 \times (4)}{5 - 4} = 15\text{-}16 = \text{-}1$$

∴ P(4, -1) divides AB in the ratio 5 : 4 externally.

Example-2 : Find the mid-point of the line segment joining two points (3, 4) and (5, 12).

**Solution:** Let A(3, 4) and B(5, 12) be the given points.

Let C(x, y) be the mid-point of AB. Using mid-point formula, we have,

$$x = \frac{3 + 5}{2} = 4$$

$$\text{and} \quad y = \frac{4 + 12}{2} = 8$$

∴ C(4, 8) are the co-ordinates of the mid-point of the line segment joining two points (3, 4) and (5, 12).

Example-3: The co-ordinates of the mid-point of a segment are (2, 3). If co-ordinates of one of the end points of the line segment are (6, 5), find the co-ordinates of the other end point.

A(x, y)   C(2, 3)   B(6, 5)
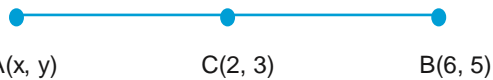
Solution: Let other end point be A(x, y)

It is given that C(2, 3) is the mid point

∴ We can write,

$$2 = \frac{x+6}{2} \quad \text{and} \quad 3 = \frac{y+5}{2}$$

or $4 = x + 6$ or $6 = y + 5$

or $x = -2$ or $y = 1$

∴ (–2, 1) are the coordinates of the other end point.

C . SLOPE OF A LINE :
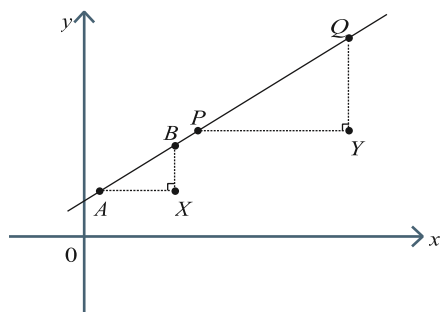
Inclination of a line : The angle made by a line with the positive direction of x-axis measured in anti clockwise .

SLOPE OR GRADIENT OF A LINE : Slope of a line is tangent of angle of inclination of the line. Generally it is denoted by **the letter 'm'. If θ is the inclination of the line then slope of the line , m= tanθ .**

SLOPE OF A LINE IF TWO POINTS ON IT ARE GIVEN :

Slope of the line-segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by ,

$$m = \frac{y2 - y1}{x_2 - x_1}$$

Example : A line passes through the points (1, 2) and (5, 10). Find its gradient .

Solution :   Slope $= \frac{10-2}{5-1} = \frac{8}{4} = 2$

Since $\tan\theta$ is not defined when $\theta = \pi/2$, the slope of a line perpendicular to the x-axis (i. e) parallel to the y-axis is not defined.

Condition of perpendicularity and parallelism :

Two non-vertical straight lines are (i) **parallel** if only if their slopes are equal and **perpendicular** if and only if **the product of their slopes is -1** .

Angle between two lines :-

If m1 and m2 are the slopes of two lines, then the tangent of the measure of the angles between them are given by

$$\tan\theta = \frac{\pm(m1 - m2)}{1 + m1m2}$$

Examples : Find the $\tan\theta$ where θ is the acute & obtuse angles between the lines having slopes 1 & ½ .

Solution: $\tan P = \frac{\pm(m2 - m1)}{1 + m1m2} = \pm\left(\frac{\frac{1}{2} - 1}{1 + \frac{1}{2}.1}\right) = \pm\frac{\left(-\frac{1}{2}\right)}{\frac{3}{2}} = \pm\frac{1}{3}$

Locus of an equation and equation of A locus :

Definition of Locus: A graph or locus of an equation in x and y is defined as the path traced by a moving point (x,y) whose co-ordinates satisfy the given equation.

Definition of equation :

The equation of a locust of a moving point (x,y) is the relation between x and y satisfied by the co-ordinates of all points of the locus and by no others .

If $P(x, y)$ be any point on the locus then it should obey the geometric condition which defines the locus. There are algebraic relation between X and Y obtained by using the geometric property represents the equation of the given locus .

STRAIGHT LINE :-

A set of points in a fixed direction is called a straight line.

Equation of a St. Line in different forms

(i) One-point form :

Equation of aline passing through the point (x_1, y_1) with slope 'm' is

$(Y - y_1) = m (X - x_1)$

(ii) Two-point form :-

Equation of the line passing through two ponts (x_1, y_1) and (x_2, y_2) is

$(Y - y_1) = \frac{(y2-y1)}{(x2-x1)} (X - x_1)$

(Iii) Slope – intercept form :

Equation of a line with slope m and whose y -intercept is c is **y=mx+ c** .

(iv) Intercepts form :

Equation of a line with x-intercept 'a' and y-intercept 'b' is given **by** $\frac{x}{a} + \frac{y}{b} = 1$

(V) Perpendicular form :

If 'p' be the length of the perpendicular from the origin to a straight line and α be the inclination of this perpendicular then the equation of the straight line is

$X \cos \alpha + y \sin \alpha = p$.

GENERAL FORM :

It may be observed that **the equation of the straight line in the above five forms are all linear in x and y** this suggests that the general linear equation in x and y

Ax + By +Cz = 0 where A, B ,C are real constants and both A and B are not 0 , a straight line.

(Vi) Equation of a line passing through one point and parallel to one line

If one line passes through a pt. (x_1, y_1) and llel to a line y= mx+c then the equation of the required line is given by

$(Y - y1) = m (X - x_1)$ (as slope of two llel lines are equal .

Vii) Equation of a line passing through one point and Perpendicular to one line

If one line passes through a point (x_1, y_1) and perpendicular to a line y = mx + c then the equation of the line is given by

$(Y - y1) = (-\frac{1}{m}) (X - x1)$

(Viii) Equation of a line passing through intersection of two lines $A_1 x + B_1 y + C1=0$ and

$A_2 x + B_2 y + C2=0$ is given by

$(A_1x + B_1y + C1) + k(A_2x + B_2y + C2) = 0$ Where k is any constant which can be found out by the given condition .

Problems :

1 . Find the equation of the line passing through the point (-3,3) and having slope -1/√3 .

Solution :   The coordinates of the given point are (- 3,3)

   slope of the line = -1/√ 3

So, equation of the line is

$(Y -3) = -\dfrac{1}{\sqrt{3}}(x+3)$

⇨ √3y -3√3 = -x -3
⇨ X +√3y +3-√3 = 0   (ans)

2. Find the equation of straight line making an intercept 3 from Y-axis and whose inclination is 135° .

Solution :

The inclination of the line is 135° . slope of the line = tan 135°= -1

y- intercept of the line= 3

so, Equation of the line is,

      y = m x + 3

⇨ Y= (-1). X +3
⇨ x+ y +3 = 0   (ans)

Example-3 Find the equation of the line passing through two points (2,-3) & (1,5) .

Solution :

Two given points are (2,-3) & (1,5) .

Equation of the line passing through above two points is

   Y- (-3) = $\dfrac{[5-(-3)]}{-1}$ (x - 2)

   ⇨ Y+3 = $\dfrac{(5+3)}{-1}$ (x-2)
   ⇨ Y+3 = -8 (x - 2)

⇨ Y+3 = -8x +16

⇨ 8x + y -13 = 0   (ans)

Example-4 Find the equation of the line which passes through the point (2, 3) and has equal intercepts on the axes .

Solution : Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$

As the intercepts are equal , a= b

So, the equation becomes

$\frac{x}{a} + \frac{y}{a} = 1 \Rightarrow \frac{x+y}{a} = 1 \Rightarrow x + y = a$

As the line passes through (2,3),

2 + 3 = a ⇒ a= 5

Therefore the required equation of the line is ,   x+ y = 5 .   (Ans)

Example-5   Find the equation of Line whose length of the perpendicular from the origin is √2 and the inclination of the perpendicular is 45° .

Solution : Here given that P= √2

 And $\alpha = 45°$

So, equation of the line is,

 X cos α + y sin α = p

⇨ Xcos 45° + y sin45° = √2

⇨ $x \cdot \frac{1}{\sqrt{2}} + y \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$

⇨ $\frac{x+y}{\sqrt{2}} = \sqrt{2}$

⇨ x + y = 2   (Ans)

Example-6 find the equation of the line which is parallel to 2x+ y-1 = 0 and passes through (1 ,6) .

Solution : As the line is llel to the line 2x+ y-1 = 0 , slope of the required line is -2 & also it passes through the point (1,6) .

So equation of the line is ,

 y-6 = -2 (x-1) ⇒ y − 6 = -2x +2

⇨ 2x + y -8 = 0   (ans)

Example-7 Find the equation of the line which is perpendicular to $x + 2y = 2$ and passes through $(2, 1)$

Solution : Slope of the given line is $(-1/2)$.

So, slope of the required line is $-\dfrac{1}{\left(-\frac{1}{2}\right)} = 2$

It passes through $(2,1)$ also ,

Hence equation of the line is ,

$\quad (y - 1) = 2(X - 2) \Rightarrow y - 1 = 2x - 4$

$\quad \Rightarrow 2x - y - 3 = 0 \quad$ (ans)

Example -8 Find the equation of the line Which passes through the point of intersection of the lines $2x - 3y = 1$ and $x + 3y = 4$ & having slope 2 .

Solution : Equation of the line passing through two pt. Of intersection of lines $2x - 3y = 1$ and $x + 3y = 4$ is

$(2x - 3y - 1) + k(x + 3y - 4) = 0$

$\quad \Rightarrow (K + 2) x + (3k - 3) y + (-4k - 1) = 0$

$\quad$ Slope of this line $= -\dfrac{(k+2)}{(3k-3)}$

$\quad$ Given that slope of this line $= 2$

$\quad$ So, $-\dfrac{(k+2)}{(3k-3)} = 2$

$\quad \Rightarrow -(k+2) = 2(3k-3) = 6k - 6$

$\quad \Rightarrow 7k = -6 + 2 = -4 \Rightarrow k = -4/7$

$\quad$ So equation of the line is ,

$\quad (-\frac{4}{7} + 2)x + \{3 \times \left(-\frac{4}{7}\right) - 3\}y - 4 \times (-\frac{4}{7}) - 1 = 0$

$\quad \Rightarrow \dfrac{-4+14}{7} x + \dfrac{-12-21}{7} y + \dfrac{16-7}{7} = 0$

$\quad \Rightarrow \dfrac{10}{7}x - \left(\dfrac{33}{7}\right) y + \dfrac{9}{7} = 0$

$\quad \Rightarrow 10x - 33y + 9 = 0 \quad (ans)$

Example-9 Find the slope and y-intercept of the line $2x - 3y + 5 = 0$

Solution : Given that $2x - 3y + 5 = 0$

$\quad \Rightarrow 3y = 2x + 5$

$\quad \Rightarrow Y = \dfrac{2}{3} x + \dfrac{5}{3}$

So , Slope of the line is 2/3 and y- intercept is 5/3 .

Example-10 **Find the value of k if the lines 2x -3y+7= 0 & x-ky+2=0 are perpendicular to each other .**

Solution: Slope of 2x-3y+7=0 , $= -\dfrac{2}{(-3)} = \dfrac{2}{3}$

Slope of x-ky+2=0 , = -1/ -k = 1/k

Given that , 2x -3y+7= 0 & x-ky+2=0 are perpendicular to each other .

So, the product of their slopes = -1

So, $\left(\dfrac{1}{k}\right)\left(\dfrac{2}{3}\right)$ = -1

⇨ K = -2/3   (ans)

Distance of a point from a line :-

The length of the Perpendicular Distance from a point (X1, y1) on the line Ax +By+ C = 0 ,

$$\mathbf{d=}\dfrac{|Ax1+By1+C|}{\sqrt{(A^2+B^2)}}$$

Corollary: The Length of the Perpendicular from the origin on the line Ax +By+ C = 0 is given by,

$$\mathbf{d} = \dfrac{|C|}{\sqrt{A^2+B^2}}$$

Example:1 Find the distance of the point (2,3) from the line 2x+ 3y -9 = 0 .

Solution: The distance of the point (2,3) from the line 2x+ 3y -9 = 0 **,**

$\mathbf{d} = \dfrac{|2.2+3.3-9|}{\sqrt{2^2+3^2}} = \dfrac{4}{\sqrt{13}}$ units

Example :2 Find the distance of the origin from the line 2x -y- 1= 0 .

Solution : $\mathbf{d} = \dfrac{|C|}{\sqrt{A^2+B^2}}$

So, here d= $\dfrac{|-1|}{\sqrt{2^2+(-1)^2}} = \dfrac{1}{\sqrt{5}}$ *units (ans)*

SHORT QUESTIONS :-

1. Find the equation of the line whose x-intercept is 3 & y- intercept is 4 . (W-19)
2. Find the value of k if the lines 2x -3y+7= 0 & x-ky+2=0 are perpendicular to each other . (W-20)
3. Find the slope and y intercept of the line 2X - 3Y + 8 =0

LONG QUESTIONS :-

1. Obtain the equation of the line passing through the point (-2,3) and perpendicular to the line $3X + 4Y - 11 = 0$. (W -19)
2. Find the equation of the line passing through the intersection of $2x$-y $-1=0$ and $3X - 4Y + 6=0$ and parallel to the line x +y- 2 =0   (S-19 , w-20)
3. Find the equation of the line passing through the $(2 - 4)$ and parallel to the line $4x+y -3=0$ (W-20)

$\approx \emptyset \approx$

Chapter- 5
Three Dimensional Geometry

Contents :

➢ **Distance formula, Section formula**
➢ **Direction cosines and direction ratios of a line, formula for angle between two lines condition of perpendicularity & parallelism**
➢ **Equation of a plane, General form**
➢ **Angle between two planes**
➢ **Problems**
➢ **Perpendicular distance of a point to a plane**
➢ **Equation of a plane passing through a point (i) llel to a plane (ii) perpendicular to a plane**

Coordinate System

The three mutually perpendicular lines in a space which divides the space into eight parts and if these perpendicular lines are the coordinate axes, then it is said to be a coordinate system.
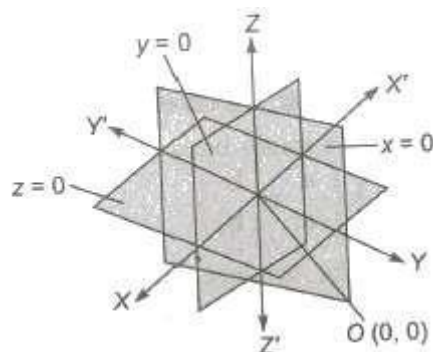
Sign Convention

Octant Coordinate	x	y	z
OXYZ	+	+	+
OX'YZ	–	+	+
OXY'Z	+	–	+
OXYZ'	+	+	–
OX'Y'Z	–	–	+
OX'YZ'	–	+	–
OXY'Z'	+	–	–
OX'Y'Z'	–	–	–

A. Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. The distance between these points is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The distance of a point $P(x, y, z)$ from origin O is

$$OP = \sqrt{x^2 + y^2 + z^2}$$

Section Formulae

(i) The coordinates of any point, which divides the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio **m : n internally** are

$$(mx_2 + nx_1 / m + n, my_2 + ny_1 / m + n, mz_2 + nz_1 / m + n)$$

(ii) The coordinates of any point, which divides the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio **m : n externally** are

$$(mx_2 - nx_1 / m - n, my_2 - ny_1 / m - n, mz_2 - nz_1 / m - n)$$

(iii) The coordinates of **mid-point** of P and Q are

$$(x_1 + x_2)/ 2 , (y_1 + y_2)/ 2, (z_1 + z_2)/ 2)$$

(iv) Coordinates of the centroid of a triangle formed with vertices $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ are

$$(x_1 + x_2 + x_3 / 3 , y_1 + y_2 + y_3 / 3, z_1 + z_2 + z_3 / 3)$$

(v) **Centroid of a Tetrahedron**

If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then its centroid G is given by

$$(x_1 + x_2 + x_3 + x_4 / 4 , y_1 + y_2 + y_3 + y_4 / 4, z_1 + z_2 + z_3 + z_4 / 4)$$
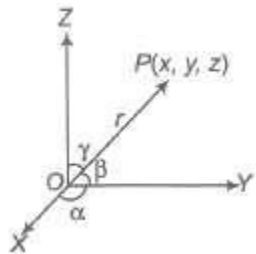
Direction Cosines & direction ratios of a Line :

If a directed line segment OP makes angle α, β and γ with OX , OY and OZ respectively, then

Cos α, cos β and cos γ are called direction cosines of OP and it is represented by l, m, n. i.e.,

$l = \cos \alpha$

$m = \cos \beta$
and $n = \cos\gamma$

If OP = r, then coordinates of OP are (lr, mr , nr)

(i) If 1, m, n are direction cosines of a vector r, then

(a) $l^2 + m^2 + n^2 = 1$

(b) Projections of r on the coordinate axes are

(c) $|r| = l|r|,\ m|r|,\ n|r|$ / $\sqrt{}$sum of the squares of projections of r on the coordinate axes

(ii)   If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, such that the direction cosines of PQ are 1, m, n. Then, $x_2 - x_1 = l|PQ|,\ y_2 - y_1 = m|PQ|,\ z_2 - z_1 = n|PQ|$

These are projections of PQ on X , Y and Z axes, respectively.

Direction ratios of a line:

If 1, m, n are direction cosines of a vector r and a b, c are three non-zero real numbers, such that

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} \ .$$

Then, we say that the direction ratios of r are proportional to **a, b,c**.

Also, we have

$$l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}} \ , \quad m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}} \ , n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$$

The signs of the d.c.s are determined according to the position of

the line with respect to the coordinate axes .

Angle Between Two Intersecting Lines :-

(iii) If θ is the angle between two lines having direction cosines $< l_1, m_1, n_1>$ & $<l_2, m_2, n_2>$

then

cos θ = $l_1 l_2$ + $m_1 m_2$ + $n_1 n_2$

Condition of parallelism & perpendicularity :-

(a) **Lines are parallel, if $l_1 / l_2 = m_1 / m_2 = n_1 / n_2$**

(b) **Lines are perpendicular, if $l_1 l_2$ + $m_1 m_2$ + $n_1 n_2$**

(v) If θ is the angle between two lines whose direction ratios are proportional to a_1, b_1, c_1

and a_2, b_2, c_2 respectively, then the angle θ between them is given by

$$\mathbf{cos\ \theta = } \quad \frac{(a1a2 + b1b2 + c1c2\ )}{(\sqrt{a1^2 + b1^2 + c1^2}\ )(\sqrt{a2^2 + b2^2 + c2^2}}$$

Lines are parallel, if $a_1 / a_2 = b_1 / b_2 = c_1 / c_2$

Lines are perpendicular, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.

(vi) The projection of the line segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ to the line having direction cosines 1, m, n is

$| (x_2 - x_1) \, l + (y_2 - y_1) \, m + (z_2 - z_1) \, n|$.

(vii) The direction ratio of the line passing through points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

proportional to $(x_2 - x_{1)}, (\, y_2 - y_{1)} \, (z_2 - z_{1)}$

 Then, direction cosines of PQ are $x_2 - x_1 / |PQ|$, $y_2 - y_1 / |PQ|$, $z_2 - z_1 / |PQ|$

 Angle Between Two Intersecting Lines :-

If (l_1, m_1, n_1) and $(l2, m_2, n_2)$ be the direction cosines of two given lines, then the angle θ between them is given by $\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

(i) The angle between any two diagonals of a cube is $\cos^{-1} (1/3)$.

(ii) The angle between a diagonal of a cube and the diagonal of a face (of the cube is $\cos^{-1} (\sqrt{2} / 3)$

Skew Lines Two straight lines in space are said to be skew lines, if they are neither parallel nor intersecting.

EXAMPLES:

1. Find the distance between the points $(3,2,1)$ &$(5,1,4)$.
    Solution:- Let P is $(3,2,1)$ & Q is $(5,1,4)$.
    Then By distance formula,

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(5-3)^2 + (1-2)^2 + (4-1)^2} = \sqrt{(4+1+9)} = \sqrt{14}$ units

Q-2 Find the coordinates of the point which divides the join of the point $(-4,1,-2)$ and $(3,8,5)$ in the ratio 5:2 .

Solution: Let P (x,y,z) be the point which divides the line joining the points $(-4,1,-2)$ and $(3,8,5)$.

Then , $x = \{5(3)+2(-4)\}/(5+2) = (15-8)/7 = 1$

$Y = (5 \times 8 + 2 \times 1)/(5+2) = 42/7 = 6$

$Z = (5 \times 5 + 2(-2))/ 5+2 = 21/7 = 3$

Hence the point is $(1,6,3)$.

Q-3 If α , β, γ be the angle which a given line makes with the positive direction of the axes then prove that $\sin^2\alpha + \sin^2\beta + \sin^2 \gamma = 2$

Solution : we know that $l^2 + m^2 + n^2 = 1$

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$

=> $3 - \sin^2\alpha - \sin^2\beta - \sin^2\gamma = 1$

=> $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3\text{-}1$

=> $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$ (proved)

Example - 4 find the distance between the points (1,-3,4) and Q(-4,1,2) .

SOLUTION :

The distance between P & Q , $PQ = \sqrt{\{(-4-1)^2 + (1+3)^2 + (2\text{-}4)(2-4)^2\}} = \sqrt{25 + 16 + 4} = \sqrt{45}$ UNITS

- NOTE : IF three points P, Q, R are collinear , then the sum of two points is equal to the third point .

Example -5 Find the coordinates of the point which divides line segment joining A(1,-2,3) and B (3,4,-5) in the ratio 2:3 (i) internally & (ii) externally .

Solution :

(i)   Let P(x,,y,z) be the point which divides the segment joining A & B internally in the ratio 2:3. Therefore

$X = \frac{2(3)+3(1)}{2+3} = 9/5$, $y = \frac{2(4)+3(-2)}{2+3} = 2/5$, $z = \frac{2(-5)+3(3)}{2+3} = \text{-}1/5$

Thus the required point is (9/5 , 2/5 , -1/5) .

(ii)   Let P(x,,y,z) be the point which divides the segment joining A & B externally in the ratio 2:3. Then

$X = \frac{2(3)-3(1)}{2-3} = \text{-}3$, $y = \frac{2(4)+3(-2)}{2+3} = \text{-}14$, $z = \frac{2(-5)+3(3)}{2+3} = 19$

Therefore the required point is (-3,-14,19)

Example-6   Find the direction cosines of a line if it makes equal angles with the coordinate axes .

Solution : As the line makes equal angles with the coordinate axes the d.c.s of the line are equal.

So, $l = m = n$

We know that $l^2 + m^2 + n^2 = 1 \Rightarrow l^2 + l^2 + l^2 = 1 \Rightarrow 3l^2 = 1 \Rightarrow l = 1/\sqrt{3}$

So, m= n = = $1/\sqrt{3}$

$\therefore$ d.c.s of the given line are $< 1/\sqrt{3}, \frac{1}{\sqrt{3}}, 1/\sqrt{3} >$.

B. <u>PLANE</u> :

A plane is a surface such that, if two points are taken on it, a straight line joining them lies wholly on the surface.
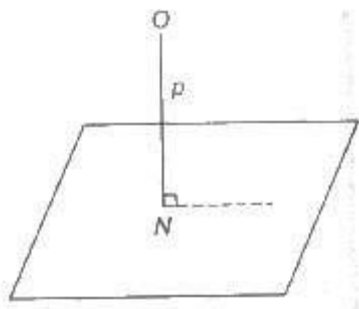
b.i General Equation of the Plane

The general equation of the first degree in x, y, z always represents a plane.

Hence, the general equation of the plane is   **ax + by + cz + d = 0**.

Where a , b, c the coefficients of x, y and z in the cartesian equation of a plane are the direction ratios of **normal to the plane**.

Normal Form of the Equation of Plane

(i) The equation of a plane, which is at a distance p from origin and the direction cosines of the normal from the origin to the plane are l, m, n is given by lx + my + nz = p.

(ii) The coordinates of foot of perpendicular N from the origin on the plane are (1p, mp, np).

Intercept Form

The intercept form of equation of plane represented in the form of

x / a + y / b + z / c = 1 where, a, b and c are intercepts on X, Y

and Z-axes, respectively.

For x intercept Put y = 0, z = 0 in the equation of the plane and obtain the value of x. Similarly, we can determine for other intercepts.
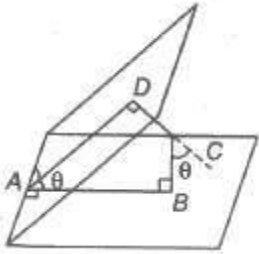
b. (ii) Angle between Two Planes

The angle between two planes is defined as the angle between the normal to them from any point.

Thus, the angle between the two planes a_1x

$+ b1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$

is equal to the angle between the normals with direction cosines

$\pm a_1 / \sqrt{\Sigma} \, a^2_1, \pm b_1 / \sqrt{\Sigma} \, a^2_1, \pm c_1 / \sqrt{\Sigma} \, a^2_1$

and $\pm a_2 / \sqrt{\Sigma} \, a^2_2, \pm b_2 / \sqrt{\Sigma} \, a^2_2, \pm c_2 / \sqrt{\Sigma} \, a^2_2$

If θ is the angle between the normals, then

$\cos \theta = \pm a_1a_2 + b_1b_2 + c_1c_2 / \sqrt{a^2_1 + b^2_1 + c^2_1} \, \sqrt{a^2_2 + b^2_2 + c^2_2}$

Condition of Parallelism and Perpendicularity of Two Planes

Two planes are parallel or perpendicular according as the normals to them are parallel or perpendicular.

Hence, the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

parallel, if $a_1 / a_2 = b_1 / b_2 = c_1 / c_2$ and perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Note : The equation of plane parallel to a given plane $ax + by + cz + d = 0$ is given by $ax + by + cz + k = 0$, where k may be determined from given conditions.

b.(iv) Perpendicular Distance from a Point to a Plane

Let the plane in the general form be $ax + by + cz + d = 0$. The distance of the point $P(x_1, y_1, z_1)$ from the plane is equal to

$$\left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$
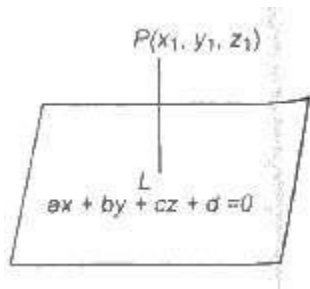
Example -7 Find the angle between the planes 2x- y+ z = 6 and x +Y + 2z = 3 .

The required angle between the given plane is the angle between their normal .

Now the d.r.s of their normal are (2,-1 , 1) and (1, 1 ,2) .

So , the d.c.s of their normal are $<\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}>$ and $<\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}>$

∴ The required angle is $cos^{-1}[(\frac{2}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}) + (-\frac{1}{\sqrt{6}})(\frac{1}{\sqrt{6}}) + (\frac{1}{\sqrt{6}})(\frac{2}{\sqrt{6}})] = cos^{-1}1/2 = \pi/3$ (ans)

Example -8 Find the distance of the point (3, 4 , 7) from the plane x + 2y − 2z =9 .

Solution : the given plane is x + 2y − 2z =9

Its distance from (3, 4 , 7) is $\left|\frac{3+2.4-2.7-9}{\sqrt{1^2+2^2+(-2)^2}}\right| = \left|\frac{-12}{\sqrt{9}}\right| = 12/3 = 4$.

If the plane is given in, normal form $lx + my + nz = p$. Then, the distance of the point $P(x_1, y_1, z_1)$ from the plane is $|lx_1 + my_1 + nz_1 - p|$.

Distance between Two Parallel Planes

If $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ be equation of two parallel planes. Then, the distance between them is

$$\left| \frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Bisectors of Angles between Two Planes

The bisector planes of the angles between the planes $a_1x +$

$b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y +$

$c_1z + d_1 / \sqrt{\Sigma a^2_1} = \pm a_2x + b_2y + c_2z + d_2 / \sqrt{\Sigma a^2_2}$

One of these planes will bisect the acute angle and the other obtuse angle between the given plane.

CHAPTER-6

SPHERE :

> **Equation of sphere at center and radius form**
> **General form of equation of a sphere**
> **Two end points of a diameter form**
> **Problems**
> **Sort questions & Long questions**

A sphere is the locus of a point which moves in a space in such a way that its distance from a fixed point always remains constant.

a. Equation of Sphere

a.(i) Equation of sphere at centre and radius form

1. In Cartesian Form The equation of the sphere with centre (a, b, c) and radius r is

$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ …….(i)

2. Equation of Sphere with centre at

origin (0,0,0) & radius 'r' is **$x^2 + y^2 + z^2 =$**

r^2

In general, we can write $x^2 + y^2 + z^2 +$

$2ux + 2vy + 2wz + d = 0$

Here, its centre is (-u, -v, -w) and radius $= \sqrt{(u^2 + v^2 + w^2 - d)}$

Important Points to be Remembered

(i) The general equation of second degree in x, y, z is $ax^2 + by^2 + cz^2 + 2hxy + 2kyz + 2lzx + 2ux + 2vy + 2wz + d = 0$

represents a sphere, if

(a) $a = b = c \; (\neq 0)$

(b) $h = k = 1 = 0$

The equation becomes $ax^2 + ay^2 + az^2 + 2ux +$

$2vy + 2wz + d - 0$ …(A)

To find its centre and radius first we make the coefficients of x^2, y^2 and z^2 each unity by

dividing throughout by a. Thus, we have $x^2 + y^2 + z^2 + (2u / a) x + (2v / a) y + (2w / a) z +$

$d / a = 0$ …..(B)

∴ Centre is $(- u / a, - v / a, - w / a)$ and

radius $= \sqrt{u^2 / a^2 + v^2 / a^2 + w^2 / a^2 - d / a} =$

$\sqrt{u^2 + v^2 + w^2 - ad} / |a|$.

(ii)   Any sphere concentric with the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $x^2 +$

$y^2 + z^2 + 2ux + 2vy + 2wz + k = 0$

(iii)   Since, $r^2 = u^2 + v^2 + w^2$ — d, therefore, the Eq. (B) represents a real sphere, if $u^2 + v^2 + w^2$ — d > 0

a(iii)Two end points of a diameter form:

 Equation of a sphere on the line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) as a diameter is

$(x – x_1) (x – x_1) + (y – y_1) (y – y_2) + (z – z_1) (z – z_2) = 0.$

(The equation of a sphere passing through four non-coplanar points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$